RADON–NIKODYM REPRESENTATIONS OF CUNTZ–KRIEGER ALGEBRAS AND LYAPUNOV SPECTRA FOR KMS STATES

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ABSTRACT. We study relations between (H,β)–KMS states on Cuntz–Krieger algebras and the dual of the Perron–Frobenius operator $\mathcal{L}^*_{-\beta H}$. Generalising the well–studied purely hyperbolic situation, we obtain under mild conditions that for an expansive dynamical system there is a one–one correspondence between (H,β)–KMS states and eigenmeasures of $\mathcal{L}^*_{-\beta H}$ for the eigenvalue 1. We then consider representations of Cuntz–Krieger algebras which are induced by Markov fibred systems, and show that if the associated incidence matrix is irreducible then these are $*$–isomorphic to the given Cuntz–Krieger algebra. Finally, we apply these general results to study multifractal decompositions of limit sets of essentially free Kleinian groups G which may have parabolic elements. We show that for the Cuntz–Krieger algebra arising from G there exists an analytic family of KMS states induced by the Lyapunov spectrum of the analogue of the Bowen–Series map associated with G. Furthermore, we obtain a formula for the Hausdorff dimensions of the restrictions of these KMS states to the set of continuous functions on the limit set of G. If G has no parabolic elements, then this formula can be interpreted as the singularity spectrum of the measure of maximal entropy associated with G.

1. INTRODUCTION

In mathematics and physics there is a long tradition of studying (H,β)–KMS (Kubo–Martin–Schwinger) states on various types of C^*–algebras of observables. Originally, these notions stem from quantum statistical mechanics, where H refers to a given self–adjoint potential function (the *energy*) which fixes the system quantum mechanically, and β admits the interpretation as the inverse of the *temperature* of the system. The general philosophy here is that macroscopic thermodynamical properties are reflected within equilibrium states depending on β, whereas microscopic quantum mechanical behaviour of the system is described by the C^*–algebra in combination with some

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time evolution, that is a *gauge action* given by a one-parameter family $\left(\alpha_H^t\right)_{t\in\mathbb{R}}$ of $*$–automorphisms depending on H.

The first goal of this paper is to give a thorough review of the correspondence between fixed points of the dual of the Perron–Frobenius operator $\mathcal{L}^*_{-\beta H}$ and (H,β)–KMS states on a Cuntz–Krieger algebra $\mathcal{O}_A$ associated with an incidence matrix A. Under mild conditions on $\mathcal{O}_A$, we collect various facts which mainly clarify the relation between the (H,β)–KMS states and the gauge action α_H^t. The novelty here is that our approach includes the case in which the underlying dynamical system is expansive, and it therefore generalises the well–studied purely hyperbolic situation to the cases in which parabolic elements can occur. More precisely, there is the well known result of Kerr/Pinzari ([20]) and Kumjian/Renault ([25]) which asserts that if A is irreducible $H>0$ then there is a one–one correspondence between the eigenmeasures of $\mathcal{L}^*_{-\beta H}$ for the eigenvalue 1 and (H,β)–KMS states. We extend this result, using gauge–invariance of KMS-states (see Fact 8), by showing that if $H\geq 0$ then there exists a bijection from the set of eigenmeasures to the set of (H,β)–KMS states which are trivial on $\{H=0\}$ (see Fact 9).

In Section 3 we then consider Markov fibred systems (Ω,m,θ,α) with an associated incidence matrix A. We show that if A is irreducible and the Radon–Nikodym derivative of the measure m is continuous on the support of m, then the associated Cuntz–Krieger C^*–algebra $\mathcal{O}_A$ admits a representation in terms of the Radon–Nikodym derivative of m (see Theorem 3.4). We will refer to this representation as the Radon–Nikodym representation of $\mathcal{O}_A$ induced by the Markov fibred system.
Finally, in Section 4 we give a finer thermodynamical analysis of KMS states associated with the Cuntz–Krieger algebra $\mathcal{O}_A$ arising from an essentially free Kleinian group G. More precisely, the final goal of this paper is to apply the above general results to complete oriented $(n+1)$–dimensional hyperbolic manifolds whose finitely generated non-elementary fundamental group G is essentially free, that is G is allowed to have parabolic elements and G has no relations other than those arising from these parabolic elements. The limit set $L(G)$ of G is the smallest closed G–invariant subset of the boundary $\mathbb{S}^n$ of $(n+1)$–dimensional hyperbolic space $\mathbb{D}^{n+1}$. This set represents an intricate fractal set hosting the complicated dynamical action of G on $\mathbb{S}^n$. The combinatoric of this action gives rise to some incidence matrix A, which then allows to represent the topological dynamics of the action by a subshift of finite type (Σ_A,θ). In particular, there exists a topologically mixing Markov map $T:L_r(G)\to L_r(G)$ on the complement $L_r(G)$ in $L(G)$ of the parabolic fixed points of G, such that T commutes with θ under the canonical coding map $\pi:(\Sigma_A,\theta)\to(L(G),T)$ (see Section 4 for the details). Functional analytic properties of the action of G on $L(G)$ can be

studied by means of the Cuntz–Krieger algebra $\mathcal{O}_A$ associated with G, and the link between $L(G)$ and $\mathcal{O}_A$ is given by the commutative algebra $\mathcal{C}\left(L(G)\right)$ of complex-valued continuous functions on $L(G)$. Note that $\mathcal{O}_A$ is the universal C^*–algebra obtained by completing the algebraic cross–product $\mathcal{C}(L(G)) \rtimes G$ with respect to the supremum of the norms of all $*$–representations on the underlying separable Hilbert space. It is well–known that in our situation here $\mathcal{O}_A$ is nuclear, simple and purely infinite ([3], [4]), and hence admits a classification by KK–theory ([36]).

By considering a special class $\{I_s\}$ of potential functions, we show that there exists a family $\{\sigma_{s(\alpha)} \mid \alpha \in (\alpha_-, \alpha_+)\}$ of $(I_{s(\alpha)}, 1)$–KMS states on $\mathcal{O}_A$, for some real analytic function $s : (\alpha_-, \alpha_+) \to \mathbb{R}$, such that (see Theorem 4.1)

$$\sigma_{s(\alpha)}(\log |T'|) = \alpha,$$

and such that for the Hausdorff dimension of the θ-invariant weak $(-I_{s(\alpha)})$–Gibbs measure $\sigma_{s(\alpha)}|_{\mathcal{C}(L(G))}$ we have

$$\dim_H \left(\sigma_{s(\alpha)}|_{\mathcal{C}(L(G))}\right) = \alpha^{-1} \cdot \sigma_{s(\alpha)}(I_{s(\alpha)}).$$

Roughly speaking, these results are obtained by computing the Lyapunov spectrum of the map T. The proof in the convex cocompact case, that is if G has no parabolic elements, shows that these results can also be interpreted in terms of multifractal analysis as the multifractal spectrum or singularity spectrum of the measure of maximal entropy arising from T. The proof for the parabolic case employs the method of inducing and here we follow closely our investigations of the dimension spectra arising from homological growth rates given in [22]. We remark that these results are based on a slight change of the usual approach which exclusively considers multiplicative pertubations of the geometric potential function $J := \log |T'|$. For this usual approach it is well–known that in the convex cocompact case the exponent of convergence $\delta(G)$ is the unique β for which there exists a $(\beta J, 1)$–KMS state. However, if there are parabolic elements then additional to the Patterson-KMS state at $\delta(G)$ there exist further $(\beta J, 1)$-KMS states for each $\beta > \delta(G)$ which all correspond to purely atomic measures concentrated on the fixed points of the parabolic transformations (see [40], [42], and for related investigations in terms of C^*–algebras for expanding dynamical systems see e.g. [31], [16], [35], [25], [17], [28], [20], [26]). In contrast to this usual approach, in this paper we also allow additive pertubations of the potential functions. More precisely, our analysis is based on potentials of the form $sJ + P(-sJ)$, where $P(-sJ)$ refers to the topological pressure of the function $-sJ$. Note that if G has parabolic elements then, in order to get the thermodynamical formalism to work, we require T–invariant versions of the measures derived from the restrictions of the KMS states to $\mathcal{C}(L(G))$. This is subject of the appendix in which we discuss a well–known

formula of Kac in the context of Markov fibred systems. Recall that the Kac formula represents a convenient method which allows to derive an invariant measure ν for the whole system from a given measure $\widetilde{\nu}$ which is invariant under some induced transformation. The appendix gives a refinement of this well–known method by showing that there exist explicit formulae which allow to compute the Radon–Nikodym derivative of ν with respect to the original transformation of the system in terms of the Radon–Nikodym derivative of $\widetilde{\nu}$ with respect to the induced transformation. These formulae might also be of interest in general infinite ergodic theory.

2. Cuntz–Krieger algebras, subshifts and KMS states

Let I refer to a given finite alphabet of cardinality at least 2, and let $A = (a_{ij})_{i,j\in I}$ be a transition matrix with entries in $\{0,1\}$ such that in each row and column there is at least one entry equal to one. The Cuntz–Krieger algebra $\mathcal{O}_A$ associated with A, as introduced in [12], is a C^*–algebra generated by partial isometries $\{S_i \mid i \in I\}$ of a separable complex Hilbert space. The algebra $\mathcal{O}_A$ is the universal C^*–algebra generated by $\{S_i \mid i \in I\}$ and satisfying the relations

$$\sum_{j\in I} S_j S_j^* = 1 \quad \text{and} \quad S_i^* S_i = \sum_{j\in I} a_{ij} S_j S_j^*, \quad \text{for all } i \in I. \tag{2.1}$$

Recall that S is a partial isometry if and only if $S = SS^*S$. Furthermore, since $\mathcal{O}_A$ is universal with respect to the relations in (2.1), we have with δ_{ij} referring to the Kronecker symbol,

$$S_i S_i^* S_j S_j^* = \delta_{ij} S_i S_i^* \text{ and } S_i^* S_i S_j = a_{ij} S_j, \quad \text{for all } i,j \in I. \tag{2.2}$$

It is well–known that the algebra $\mathcal{O}_A$ is uniquely determined up to isomorphism by the relations in (2.1) (see [12, Theorem 2.13]). Furthermore, if A is irreducible then $\mathcal{O}_A$ is simple, which means that each closed two–sided ideal of $\mathcal{O}_A$ is trivial (see [12, Theorem 2.14]).

From a dynamical point of view, the transition matrix A gives rise to a subshift of finite type (Σ_A, θ), where

$$\begin{aligned} \Sigma_A &:= \left\{ (w_0 w_1 w_2 \ldots) \in I^{\mathbb{N}} \mid a_{w_i w_{i+1}} = 1 \text{ for all } i \in \mathbb{N}_0 \right\}, \\ \theta &: \Sigma_A \to \Sigma_A, (w_0 w_1 \ldots) \mapsto (w_1 w_2 \ldots). \end{aligned}$$

The space Σ_A is a compact metric space with respect to the metric given by

$$d\left((w_0 w_1 \ldots), (v_0 v_1 \ldots)\right) := 2^{-\min\{i \mid w_i = v_i\}}, \text{ for } (w_0 w_1 \ldots), (v_0 v_1 \ldots) \in \Sigma_A.$$

Also, it is well-known that θ is uniformly expanding with respect to this metric (see e.g. [14]). Let $\mathcal{W} := \bigcup_{n=1}^{\infty} \mathcal{W}^n$ refer to the set of finite, admissible words, for

$$\mathcal{W}^n := \{(w_0 w_1 \dots w_{n-1}) \in I^n \mid a_{w_i w_{i+1}} = 1 \text{ for } i = 0, 1, \dots n-1\}.$$

By defining for each $n \in \mathbb{N}$ and $w = (w_0 w_1 \dots w_{n-1}) \in \mathcal{W}^n$,

$$[w] := \left\{ \left(w_0' w_1' \dots\right) \in \Sigma_A \mid w_i = w_i' \text{ for } i = 0, \dots, n-1 \right\},$$

we have that $\{[w] \mid w \in \mathcal{W}^n\}$ is a partition of Σ_A consisting of closed and open sets. The link between Σ_A and $\mathcal{O}_A$ is then given by the commutative algebra $\mathcal{C}(\Sigma_A)$ of continuous, complex-valued functions on Σ_A. With $\mathbf{1}_{[w]}$ referring to the indicator function, we have that the algebra generated by $\left\{\mathbf{1}_{[w]} \mid w \in \mathcal{W}\right\}$ is dense in $\mathcal{C}(\Sigma_A)$. Therefore, the assignment $\psi\left(\mathbf{1}_{[w]}\right) := S_w S_w^*$ admits an extension to an isomorphism ψ from $\mathcal{C}(\Sigma_A)$ to the commutative subalgebra generated by $\{(S_w S_w^*) \mid w \in \mathcal{W}\}$, where $S_w := S_{w_0} S_{w_1} \cdots S_{w_{n-1}}$ for $w = (w_0 w_1 \dots w_{n-1})$. Also, combining this with the second relation in (2.1), one immediately verifies that $S_w^* S_w = \psi\left(\mathbf{1}_{\theta^n [w]}\right)$, for all $w \in \mathcal{W}^n$ and $n \in \mathbb{N}$. For ease of notation, throughout we will not distinguish between $f \in \mathcal{C}(\Sigma_A)$ and $\psi(f) \in \mathcal{O}_A$.

The following proposition gives some of the most important basic rules for the calculus within $\mathcal{O}_A$. In here, $\mathcal{F}_A$ refers to the algebra of all finite linear combinations of words in the generators S_j, S_j^*. Note that $\mathcal{F}_A$ is norm dense in $\mathcal{O}_A$. Moreover, $\tau_w : \theta^n([w]) \to [w]$ denotes the inverse branch of $\theta^n|_{[w]}$, for $w \in \mathcal{W}^n$ and $n \in \mathbb{N}$.

Proposition 2.1. *For the Cuntz–Krieger algebra $\mathcal{O}_A$, the following holds.*

(1) *Each $X \in \mathcal{F}_A$ can be written as a linear combination of terms of the form $S_v S_w^*$, for $v, w \in \mathcal{W}$.*
(2) $S_v^* S_w = \delta_{v,w} \cdot S_v^* S_v$, *for all $v, w \in \mathcal{W}^n$ and $n \in \mathbb{N}$.*
(3) $S_{v_0 \dots v_{n-1}}^* S_{v_0 \dots v_{n-1}} = S_{v_{n-1}}^* S_{v_{n-1}}$, *for all $(v_0 \dots v_{n-1}) \in \mathcal{W}^n$ and $n \in \mathbb{N}$.*
(4) $S_v^* f S_w = \delta_{v,w} \cdot f \circ \tau_v$, *for all $v, w \in \mathcal{W}^n$, $n \in \mathbb{N}$ and $f \in \mathcal{C}(\Sigma_A)$. In particular, we hence have $S_v^* f S_w \in \mathcal{C}(\Sigma_A)$ and $S_v^* f S_v = S_v^* S_v \cdot f \circ \tau_v$.*
(5) $S_v f S_v^* = \mathbf{1}_{[v]} \cdot f \circ \theta^n$, *for all $v \in \mathcal{W}^n$ and $n \in \mathbb{N}$. In particular, we hence have $S_v f S_v^* \in \mathcal{C}(\Sigma_A)$ and $S_v f S_v^* = S_v S_v^* \cdot f \circ \theta^n$.*
(6) *For all $v \in \mathcal{W}^n$, $n \in \mathbb{N}$ and $f \in \mathcal{C}(\Sigma_A)$, the following holds.*
    (a) $S_v f = (f \circ \theta^n) S_v$ *and* $f S_v = S_v (f \circ \tau_v \cdot \mathbf{1}_{\theta^n [v]})$.
    (b) $f S_v^* = S_v^* (f \circ \theta^n)$ *and* $S_v^* f = (f \circ \tau_v \cdot \mathbf{1}_{\theta^n [v]}) S_v^*$.

Proof. For (1), (2) and (3) we refer to [12, Lemma 2.1, 2.2]. For the proof of (4), we consider without loss of generality $v = (v_0 \dots v_{n-1})$, $w = (w_0 \dots w_{n-1}) \in \mathcal{W}^n$ and $u = (u_0 \dots u_{m-1}) \in \mathcal{W}^m$, for $n < m$. Clearly, by (2) we have that if either $v \neq (u_0 \dots u_{n-1})$ or $w \neq (u_0 \dots u_{n-1})$, then $S_v^* S_u S_u^* S_w = 0$. Otherwise, that is if

$v = w = (u_0 \dots u_{n-1})$, we obtain from (2) and (3),

$$\begin{aligned} S_v^* S_u S_u^* S_w &= S_v^* S_{u_0 \dots u_{n-1}} S_{u_n \dots u_{m-1}} S_{u_n \dots u_{m-1}}^* S_{u_0 \dots u_{n-1}}^* S_w \\ &= S_{u_{n-1}}^* S_{u_{n-1}} S_{u_n \dots u_{m-1}} S_{u_n \dots u_{m-1}}^* S_{u_{n-1}}^* S_{u_{n-1}} \\ &= \mathbf{1}_{\theta^n[u]} = \mathbf{1}_{[u]} \circ \tau_v. \end{aligned}$$

Since $\{\mathbf{1}_w \mid w \in \mathcal{W}\}$ is dense in $\mathcal{C}(\Sigma_A)$, the result in (4) follows. The assertion in (5) follows by similar means.

For the proof of (6), we consider without loss of generality $\mathbf{1}_{[w]} = S_w S_w^* \in \mathcal{C}(\Sigma_A)$, for $w = (w_0 \dots w_{m-1}) \in \mathcal{W}$. Since $S_w S_w^* S_j S_j^* \mathbf{1}_{[w]} \mathbf{1}_{[j]} = \delta_{w_0 j} \mathbf{1}_{[w]}$, we have for $v = (v_0 \dots v_{n-1}) \in \mathcal{W}^n$ with $a_{v_{n-1} w_0} = 1$, using (2.1), (3) and (5),

$$S_v S_w S_w^* = S_v S_w S_w^* \sum_{j \in I} a_{v_{n-1},j} S_j S_j^* = S_v \mathbf{1}_{[w]} S_v^* S_v = (\mathbf{1}_{[w]} \circ \theta^n) S_v.$$

This gives the first part in (a). For the second part note that

$$f S_v = f S_v S_v^* S_v = f \cdot \mathbf{1}_{[v]} S_v = (f \circ \tau_v \cdot \mathbf{1}_{\theta^n[v]}) \circ \theta^n S_v.$$

Therefore, using the first part, the assertion follows. The statements in (b) are immediate consequences of (a). □

Throughout, let $H = H^* \in \mathcal{C}(\Sigma_A)$ be always a given self–adjoint potential function. Then the *Perron–Frobenius operator* $\mathcal{L}_H : \mathcal{C}(\Sigma_A) \to \mathcal{C}(\Sigma_A)$ associated with H is defined in dynamical terms as follows (see [37]). For $f \in \mathcal{C}(\Sigma_A)$ and $x \in \Sigma_A$, let

$$(\mathcal{L}_H(f))(x) := \sum_{y \in \theta^{-1}(\{x\})} e^{H(y)} f(y).$$

Using Proposition 2.1 (4), we obtain that $\mathcal{L}_H$ can be expressed in algebraic terms in the following way.

$$\mathcal{L}_H(f) = \sum_{j \in I} S_j^* e^H f S_j. \tag{2.3}$$

Definition 2.2. For $t, \beta \in \mathbb{R}$, we define the following.

- A *gauge action* is a $*$–automorphism $\alpha_H^t : \mathcal{O}_A \to \mathcal{O}_A$ given by the extension to $\mathcal{O}_A$ of
$$S_j \mapsto \alpha_H^t S_j := e^{itH} S_j, \text{ for each } j \in I.$$
- A (H, β)–*KMS state* is a state σ on $\mathcal{O}_A$ such that for each pair X, Y in a norm–dense subset of $\mathcal{O}_A$, there exists an analytic function $F_{X,Y} : \{z \in \mathbb{C} \mid 0 \leq \mathfrak{Im}(z) \leq \beta\} \to \mathbb{C}$ such that
$$F_{X,Y}(t) = \sigma\left(X \alpha_H^t(Y)\right) \text{ and } F_{X,Y}(t + i\beta) = \sigma\left(\alpha_H^t(Y) X\right).$$

Recall that a state σ on a C^*–algebra $\mathcal{A}$ is by definition a linear functional for which $\|\sigma\| = 1$ and $\sigma(X^*X) \geq 0$, for all $X \in \mathcal{A}$. Also note that, since α_H^t is a $*$–automorphism, we have for all $v = (v_0 \cdots v_{m-1})$, $w = (w_0 \cdots w_{n-1}) \in \mathcal{W}$, and $t \in \mathbb{R}$,

$$\begin{aligned}
\alpha_H^t\left(S_v S_w^*\right) &= e^{\mathrm{i}tH} S_{v_0} \cdots e^{\mathrm{i}tH} S_{v_{m-1}} S_{w_{n-1}}^* e^{-\mathrm{i}tH} \cdots S_{w_0}^* e^{-\mathrm{i}tH} \\
&= e^{\mathrm{i}t\sum_{k=0}^{m-1} H\circ\theta^k} S_v S_w^* e^{-\mathrm{i}t\sum_{k=0}^{n-1} H\circ\theta^k}.
\end{aligned} \tag{2.4}$$

In here, the final equality is a consequence of Proposition 2.1 (6). In order to consider analytic continuations of the gauge action, we require the following concept of analyticity of [10, Definition 2.5.20].

Definition 2.3. An element $X \in \mathcal{O}_A$ is called α_H^t*–analytic* if there exists a positive number λ and a map $f_X : D_\lambda := \{z \in \mathbb{C} \mid |\mathfrak{Im}(z)| < \lambda\} \to \mathcal{O}_A$ such that the following holds.

1. $f_X(t) = \alpha_H^t(X)$, for all $t \in \mathbb{R}$.
2. The function $\eta \circ f_X : D_\lambda \to \mathbb{C}$ is analytic, for each map η in the topological dual $(\mathcal{O}_A)'$ of $\mathcal{O}_A$.

Fact 1 (α_H^t**-analyticity).** *The gauge action admits a unique continuation to all of $\mathbb{C}$ in the following way. For each $X = S_v S_w^*$ with $m, n \in \mathbb{N}$, $v \in \mathcal{W}^m$ and $w \in \mathcal{W}^n$, there exists a unique continuation of $\alpha_H^t(X)$ such that*

$$\alpha_H^z(X) = e^{\mathrm{i}z\sum_{k=0}^{m-1} H\circ\theta^k} X e^{-\mathrm{i}z\sum_{k=0}^{n-1} H\circ\theta^k}, \text{ for all } z \in \mathbb{C}.$$

In particular, we hence have that each $Y \in \mathcal{F}_A$ is α_H^t–analytic with respect to $D_\lambda = \mathbb{C}$.

Proof. Note that if $X \in \mathcal{O}_A$ is α_H^t–analytic and $\eta \in (\mathcal{O}_A)'$, then analyticity of $\eta \circ f_X$ together with Definition 2.3 (1) immediately implies that the function $\eta \circ f_X$ does not depend on the special choice of f_X. Combining this with the fact that $\mathcal{O}_A$ is a Hilbert space and hence is reflexive, the uniqueness of f_X follows. For the remaining assertions, first note that by [10, Proposition 2.5.21] we have that the statement in Definition 2.3 (2) is equivalent to the fact that for each $z \in D_\lambda$ the following limit exists, where the limit is taken with respect to the norm in $\mathcal{O}_A$.

$$\lim_{w\to z} \frac{f_X(z) - f_X(w)}{z - w}.$$

Hence, in order to show that each $Y \in \mathcal{F}_A$ is α_H^t–analytic, it is sufficient to show that the above limit exists, for $z \in \mathbb{C}$ and $X = S_v S_w^*$ with $v \in \mathcal{W}^m$, $w \in \mathcal{W}^n$ and

$$f_{S_v S_w^*}(z) := e^{\mathrm{i}z\sum_{k=0}^{m-1} H\circ\theta^k} S_v S_w^* e^{-\mathrm{i}z\sum_{k=0}^{n-1} H\circ\theta^k}.$$

Indeed, for $g := \sum_{k=0}^{m-1} H \circ \theta^k - \sum_{k=0}^{n-1} H \circ \theta^k \circ \tau_w \circ \theta^m$ we have by Proposition 2.1 (6) that $f_{S_v S_w^*}(z) = e^{\mathrm{i}zg} S_v S_w^*$. Hence, it follows for $z, \varepsilon \in \mathbb{C}$,

$$\lim_{\varepsilon \to 0} \frac{f_{S_v S_w^*}(z) - f_{S_v S_w^*}(z+\varepsilon)}{\varepsilon} = \left(\lim_{\varepsilon \to 0} \frac{1 - e^{\mathrm{i}\varepsilon g}}{\varepsilon} \right) \cdot e^{\mathrm{i}zg} S_v S_w^* = -\mathrm{i}g \cdot f_{S_v S_w^*}(z).$$

This shows that $S_v S_w^*$ is α_H^t–analytic with respect to $D_\lambda = \mathbb{C}$, which then clearly also holds for each $Y \in \mathcal{F}_A$. Since, as we have seen above, $f_X(z)$ is uniquely determined for all $z \in \mathbb{C}$, we can now define $\alpha_H^z(X) := f_X(z)$, which then finishes the proof. □

Note that for $z \in \mathbb{C} \setminus \mathbb{R}$ the continuation α_H^z of the gauge action α_H^t is no longer a $*$–automorphism.

Fact 2 (**KMS condition**). *For an (H, β)–KMS state σ on $\mathcal{O}_A$ with $\beta \in \mathbb{R} \setminus \{0\}$, we have*

$$\sigma(XY) = \sigma(Y \alpha_H^{\mathrm{i}\beta}(X)), \text{ for all } X, Y \in \mathcal{F}_A.$$

Proof. A similar argument as in the proof of Fact 1 shows that the assignment $z \mapsto \sigma(Y \alpha_H^z(X))$ gives rise to an analytic function on $\mathbb{C}$, for each (H, β)–KMS state σ and each $X, Y \in \mathcal{F}_A$. This implies $\sigma(Y \alpha_H^z(X)) = F_{Y,X}(z)$, for all $z \in \mathbb{C}$ for which $0 \leq \mathfrak{Im}(z) \leq \beta$. Using Definition 2.2 (2), it therefore follows that

$$\sigma(XY) = F_{Y,X}(\mathrm{i}\beta) = \sigma(Y f_X(i\beta)) = \sigma(Y \alpha_H^{\mathrm{i}\beta}(X)).$$

□

We remark that the proof of the previous fact in particular shows that

$$\sigma(\alpha_H^t(X)Y) = \sigma(Y \alpha_H^{t+\mathrm{i}\beta}(X)), \text{ for all } t \in \mathbb{R} \text{ and } X, Y \in \mathcal{F}_A.$$

For a KMS state we also immediately obtain the following fact (see e.g. [11, Proposition 5.3.3]).

Fact 3 (**State invariance**). *For an (H, β)–KMS state σ on $\mathcal{O}_A$ with $\beta \in \mathbb{R} \setminus \{0\}$, we have*

$$\sigma\left(\alpha_H^z X\right) = \sigma\left(X\right), \text{ for all } z \in \mathbb{C} \text{ and } X \in \mathcal{O}_A.$$

Let us collect further important observations concerning the thermodynamical formalism for KMS states. The following fact is adopted from [20, Section 7], and we include a proof for convenience.

Fact 4 (**Centraliser**). *For an (H, β)–KMS state σ on $\mathcal{O}_A$, we have*

$$\sigma(Xf) = \sigma(fX), \text{ for all } f \in \mathcal{C}\left(\Sigma_A\right), X \in \mathcal{O}_A.$$

Proof. Using (2.4) and Proposition 2.1 (6), we obtain for all $w \in \mathcal{W}$ and $t \in \mathbb{R}$,

$$\alpha_H^z (S_w S_w^*) = S_w S_w^*.$$

Since $\{S_w S_w^* : w \in \mathcal{W}\}$ is dense in $\mathcal{C}(\Sigma_A)$, it follows that $\alpha_H^z(f) = f$, for all $f \in \mathcal{C}(\Sigma_A)$. Using Fact 2, we hence have

$$\sigma(Xf) = \sigma(X\alpha_H^{\mathrm{i}\beta}(f)) = \sigma(fX).$$

□

Fact 5 (Faithfulness). *Let A be irreducible, and let σ be an (H,β)–KMS state. We then have for all $X \in \mathcal{O}_A$,*

$$\sigma(X^*X) = 0 \text{ if and only if } X = 0.$$

Proof. First, note that the set

$$\mathcal{I} := \{Y \in \mathcal{O}_A \mid \sigma(Y^*Y) = 0\}$$

is a left ideal of $\mathcal{O}_A$. This follows since, using the estimate $|\sigma(X^*YX)| \leq \sigma(X^*X)\|Y\|$ (see [10, Proposition 2.3.11]), we have for $Y \in \mathcal{I}$ and $X \in \mathcal{O}_A$,

$$0 \leq \sigma((XY)^* XY) \leq \|X\|^2 \sigma(Y^*Y) = 0.$$

Also, since σ is continuous, $\mathcal{I}$ is closed. Using the KMS condition and the Cauchy–Schwarz inequality for σ (see [10, Lemma 2.3.10] and proof of Fact 8), we obtain

$$\begin{aligned} |\sigma(X^*Y^*YX)|^2 &= \left|\sigma\left(\alpha_H^{-\mathrm{i}\beta}(X) X^*Y^*Y\right)\right|^2 \\ &\leq \sigma\left(\left(\alpha_H^{-\mathrm{i}\beta}(X) X^*\right)^* \alpha_H^{-\mathrm{i}\beta}(X) X^*\right) \sigma(Y^*YY^*Y) \\ &\leq \sigma\left(\left(\alpha_H^{-\mathrm{i}\beta}(X) X^*\right)^* \alpha_H^{-\mathrm{i}\beta}(X) X^*\right) \|Y\|^2 \sigma(Y^*Y) = 0. \end{aligned}$$

Since $\mathcal{O}_A$ is simple if A is irreducible (see [12, Theorem 2.14]) and since $1 \notin \mathcal{I}$, it follows that $\mathcal{I} = 0$. □

Fact 6 (β - Conformality). *For an (H,β)–KMS state σ on $\mathcal{O}_A$ and for all $v \in \mathcal{W}^m$, $w \in \mathcal{W}^n$ with $m,n \in \mathbb{N}$ such that $vw \in \mathcal{W}^{n+m}$ we have*

(1) $\sigma(S_{vw}S_{vw}^*) = \sigma\left(e^{-\beta\sum_{k=0}^{m-1} H\circ\theta^k\circ\tau_v} S_w S_w^*\right)$,

(2) $\sigma(S_w S_w^*) = \sigma\left(e^{\beta\sum_{k=0}^{m-1} H\circ\theta^k} S_{vw} S_{vw}^*\right)$.

Proof. By the KMS property, Fact 1 and Proposition 2.1 (6), it follows that

$$\begin{aligned}\sigma\left(S_{vw}S_{vw}^{*}\right) &= \sigma\left(S_{w}S_{w}^{*}S_{v}^{*}\alpha_{H}^{\mathrm{i}\beta}\left(S_{v}\right)\right)=\sigma\left(S_{w}S_{w}^{*}S_{v}^{*}e^{-\beta\sum_{k=0}^{m-1}H\circ\theta^{k}}S_{v}\right)\\ &= \sigma\left(S_{w}S_{w}^{*}S_{v}^{*}S_{v}e^{-\beta\sum_{k=0}^{m-1}H\circ\theta^{k}\circ\tau_{v}}\right)\\ &= \sigma\left(S_{w}S_{w}^{*}e^{-\beta\sum_{k=0}^{m-1}H\circ\theta^{k}\circ\tau_{v}}\right).\end{aligned}$$

The second assertion follows by precisely the same means. □

We remark that Fact 6 has the following immediate implication for the measure associated with σ by the Riesz representation theorem. Namely, with $m_{\sigma}:=\sigma|_{\mathcal{C}(\Sigma_{A})}$ referring to this measure, the second statement in Fact 6 immediately gives

$$\log\frac{dm_{\sigma}\circ\theta^{n}}{dm_{\sigma}}=\beta\sum_{k=0}^{n-1}H\circ\theta^{k},\text{ for each }n\in\mathbb{N}.$$

For the following we recall the definitions of a Gibbs measure and of a weak Gibbs measure, which we have adapted to our situation here (see [21], [43]).

Definition 2.4. Let μ a Borel probability measure on Σ_A for which there exists $f\in\mathcal{C}\left(\Sigma_{A}\right)$ and a sequence $\left(b_{n}\right)_{n\in\mathbb{N}}$ of positive numbers such that, for all $n\in\mathbb{N}$, $w\in\mathcal{W}^{n}$ and $x\in[w]$,

$$e^{-b_{n}}\leq\frac{\mu([w])}{e^{\sum_{k=0}^{n-1}f\circ\theta^{k}(x)}}\leq e^{b_{n}}.$$

(1) The measure μ is called *f–Gibbs measure* if the sequence (b_n) is constant.
(2) The measure μ is called *weak f–Gibbs measure* if $\lim_{n\to\infty}b_{n}/n=0$.

Recall that if f is a strictly negative Hölder continuous potential function then the Perron–Frobenius–Ruelle theorem (see e.g. [7]) implies that there exists a unique f–Gibbs measure. Also, note that the concept of a weak f–Gibbs measure is slightly more general than the concept of an f–Gibbs measure. Namely, if for instance $f\in\mathcal{C}(\Sigma_{A})$ is not Hölder continuous and $f\leq0$ such that $f(x)=0$ for at most finitely many $x\in\Sigma_{A}$, then it is possible that there exists a weak f–Gibbs measure which is not an f–Gibbs measure. In particular, this weak f–Gibbs measure is not necessarily unique (see [21], [43]). Next recall that the topological pressure of $f\in\mathcal{C}\left(\Sigma_{A}\right)$ is given by

$$P(f):=\lim_{n\to\infty}\frac{1}{n}\log\sum_{w\in\mathcal{W}^{n}}\exp\left(\sup_{x\in[w]}\sum_{k=0}^{n-1}f\circ\theta^{k}(x)\right).$$

One immediately verifies that for a potential $f\in\mathcal{C}\left(\Sigma_{A}\right)$ for which there exists a weak f–Gibbs measure, we necessarily have that $P(f)=0$. For the following fact note that in [21] it was shown that if μ is contained in the set $\mathrm{Fix}\left(\mathcal{L}_{f}^{*}\right)$ of probability measures

ν which are eigenmeasures of the dual $\mathcal{L}_f^*$ of the Perron–Frobenius operator for the eigenvalue 1, that is for which $\mathcal{L}_f^*\nu = \nu$, then μ is a weak f–Gibbs measure. In this situation we then have

$$\frac{d\mu \circ \theta}{d\mu} = e^{-f}.$$

Note that a weak f–Gibbs measure has no atoms if f is strictly negative.

Fact 7 (**$\mathcal{L}^*$–invariance**). *For an (H, β)–KMS state σ on $\mathcal{O}_A$ with $\beta \in \mathbb{R} \setminus \{0\}$, we have*

$$\sum_{j \in I} \sigma\left(S_j^* \mathrm{e}^{-\beta H} X S_j\right) = \sigma(X), \text{ for all } X \in \mathcal{O}_A.$$

We then in particular have that

$$m_\sigma := \sigma|_{\mathcal{C}(\Sigma_A)} \in \mathrm{Fix}\left(\mathcal{L}^*_{-\beta H}\right),$$

and hence that m_σ is a weak $(-\beta H)$–Gibbs measure.

Proof. Since $\alpha_H^{-\mathrm{i}\beta}\left(S_j^*\right) = S_j^* \mathrm{e}^{-\beta H}$ for all $j \in I$, a combination of (2.1) and Definition 2.2 (2) gives that

$$\begin{aligned} \sigma(X) &= \sum_{j \in I} \sigma\left(X S_j S_j^*\right) = \sum_{j \in I} \sigma\left(X S_j \alpha_H^{\mathrm{i}\beta} \circ \alpha_H^{-\mathrm{i}\beta}(S_j^*)\right) \\ &= \sum_{j \in I} \sigma\left(S_j^* \mathrm{e}^{-\beta H} X S_j\right). \end{aligned}$$

The assertion $\mathcal{L}^*_{-\beta H} m_\sigma = m_\sigma$ is an immediate consequence of (2.3). □

The following fact gives a generalisation of [20, Lemma 7.5]. In here, we have put $\{H = 0\} := \{x \in \Sigma_A \mid H(x) = 0\}$, which is a closed subset of Σ_A.

Fact 8 (**Gauge invariance**). *For an (H, β)–KMS state σ on $\mathcal{O}_A$, we have*

$$\sigma(S_v S_w^*) = 0, \text{ for all } v, w \in \mathcal{W}^n,\ v \neq w.$$

Furthermore, if $\beta \in \mathbb{R} \setminus \{0\}$ and $H \geq 0$ such that $m_\sigma(\{H = 0\}) = 0$, then

$$\sigma(S_v S_w^*) = 0, \text{ for all } v, w \in \mathcal{W},\ v \neq w.$$

In particular, we hence have $\sigma(X) = 0$, for all X in the closure of the vector space generated by $\{S_v S_w^* \mid v \in \mathcal{W}^m, w \in \mathcal{W}^n, m \neq n\}$.

Proof. Let $v = (v_0 \dots v_{m-1})$, $w = (w_0 \dots w_{n-1}) \in \mathcal{W}$ be given. We then have by Fact 2 and Proposition 2.1 (6),

$$\begin{aligned} \sigma(S_v S_w^*) &= \sigma(S_w^* \alpha_H^{\mathrm{i}\beta} S_v) = \sigma\left(S_w^* e^{-\beta \sum_{k=0}^{m-1} H \circ \theta^k} S_v\right) \\ &= \sigma\left(e^{-\beta \sum_{k=0}^{m-1} H \circ \theta^k \circ \tau_w} \mathbf{1}_{\theta^n[w]} S_w^* S_v\right). \end{aligned}$$

For $n = m$ Proposition 2.1 (2) then gives the first assertion. For $n \neq m$ note that Proposition 2.1 (2) implies that $\sigma(S_v S_w^*) = 0$ if either $m > n$ and $(v_0 \dots v_{n-1}) \neq w$ or $m < n$ and $(w_0 \dots w_{m-1}) \neq v$. For the remaining statements we only consider the case $m < n$. The other cases can be dealt with in an analogous way. Using Facts 1, 3 and 4, we obtain for each $t \in \mathbb{R}$ and v, w such that $(w_0 \dots w_{m-1}) = v$,

$$\begin{aligned}\sigma(S_v S_w^*) &= \sigma(\alpha_H^{\mathrm{it}}(S_v S_w^*)) = \sigma\left(e^{-t\sum_{k=0}^{m-1} H\circ\theta^k} S_v S_w^* e^{t\sum_{k=0}^{n-1} H\circ\theta^k}\right)\\ &= \sigma\left(S_v S_w^* e^{t\sum_{k=m}^{n-1} H\circ\theta^k}\right)\end{aligned}$$

Recall that by the Cauchy–Schwartz inequality $|\sigma(X^*Y)|^2 \leq \sigma(X^*X)\sigma(Y^*Y)$ for all $X, Y \in \mathcal{O}_A$ (see [10, Lemma 2.3.10]). Hence, by the monotone convergence theorem, we have

$$\begin{aligned}|\sigma(S_v S_w^*)|^2 &= \left|\sigma\left(S_v S_w^* e^{t\sum_{k=m}^{n-1} H\circ\theta^k}\right)\right|^2\\ &\leq \sigma\left(S_v S_w^* S_w S_v^*\right)\sigma\left(e^{2t\sum_{k=m}^{n-1} H\circ\theta^k}\right)\\ &= \sigma\left(S_v S_w^* S_w S_v^*\right) m_\sigma\left(e^{2t\sum_{k=m}^{n-1} H\circ\theta^k}\right)\\ &\to \sigma\left(S_v S_w^* S_w S_v^*\right) m_\sigma\left(\mathbf{1}_{\bigcap_{k=m}^{n-1}\{H\circ\theta^k=0\}}\right), \text{ for } t \to -\infty.\end{aligned}$$

Since $m_\sigma\left(\mathbf{1}_{\bigcap_{k=m}^{n-1}\{H\circ\theta^k=0\}}\right) \leq m_\sigma\left(\mathbf{1}_{\{H\circ\theta^m=0\}}\right)$, it is now sufficient to show that $m_\sigma\left(\mathbf{1}_{\{H\circ\theta^m=0\}}\right) = 0$, for all $m \in \mathbb{N}$. Indeed, using $\mathcal{L}_{-\beta H}^* m_\sigma = m_\sigma$ (see Fact 7), Proposition 2.1 (4) and the assumption $m_\sigma\left(\mathbf{1}_{\{H=0\}}\right) = 0$, it follows

$$\begin{aligned}m_\sigma\left(\mathbf{1}_{\{H\circ\theta^m=0\}}\right) &= m_\sigma\left(\mathbf{1}_{\{H=0\}}\circ\theta^m\right) = m_\sigma\left(\mathcal{L}_{-\beta H}^m(\mathbf{1}_{\{H=0\}}\circ\theta^m)\right)\\ &= m_\sigma\left(\sum_{v\in\mathcal{W}^m} S_v^* e^{-\beta H}\mathbf{1}_{\{H=0\}}\circ\theta^m S_v\right)\\ &= m_\sigma\left(\sum_{v\in\mathcal{W}^m} S_v^* e^{-\beta H} S_v \mathbf{1}_{\{H=0\}}\circ\theta^m\circ\tau_v\right)\\ &= m_\sigma\left(\mathbf{1}_{\{H=0\}}\mathcal{L}_{-\beta H}^m\mathbf{1}\right) = 0.\end{aligned}$$

□

Fact 9 (KMS vs. Gibbs). *Let A be irreducible and $\beta \in \mathbb{R}\setminus\{0\}$. Then the assignment $\Theta(\sigma) := m_\sigma := \sigma|_{\mathcal{C}(\Sigma_A)}$ gives rise to a well–defined surjective map*

$$\Theta : \{\sigma \mid \sigma \textit{ is a } (H,\beta)\text{–}\textit{KMS state}\} \quad\to\quad \mathrm{Fix}\left(\mathcal{L}_{-\beta H}^*\right).$$

Furthermore, if $H \geq 0$ then the restriction of Θ to $\mathcal{S}_0 := \{\sigma \mid \sigma$ is a (H,β)–KMS state such that $m_\sigma(\{H=0\}) = 0\}$ is a bijection onto its image.

Proof. By the Riesz representation theorem, images under Θ are in fact Borel probability measures, which combined with Fact 6 gives that Θ is well–defined. In order to show surjectivity of Θ, we have to construct a state on $\mathcal{O}_A$ from a given Borel measure $\mu \in \mathrm{Fix}\left(\mathcal{L}^*_{-\beta H}\right)$. For this we define for each $v, w \in \mathcal{W}$,

$$P(S_v S_w^*) := \begin{cases} 0 & : \quad w \neq v \\ \mathbf{1}_{[v]} & : \quad w = v. \end{cases}$$

This can be extended to a linear map from $\mathcal{F}_A \to \mathcal{C}(\Sigma_A)$ with $\|P(A)\| \leq \|A\|$ for each $A \in \mathcal{F}_A$ and $\|P(A)\| = \|A\|$ for $A \in \mathcal{F}_A \cap \mathcal{C}(\Sigma_A)$. Hence, P admits a further extension to a continuous and linear projection $P : \mathcal{O}_A \to \mathcal{C}(\Sigma_A)$. This allows to define the functional σ_μ by

$$\sigma_\mu(A) := \int P(A) d\mu \quad \text{ for } A \in \mathcal{O}_A.$$

In particular, σ_μ is a positive linear functional with $\|\sigma_\mu\| = 1$, and hence is a state. In order to see that σ_μ satisfies the KMS condition if $\mu \in \mathrm{Fix}\left(\mathcal{L}^*_{-\beta H}\right)$, it is sufficient to show that for each $w, v, w', v' \in \mathcal{W}$,

$$\sigma_\mu(S_v S_w^* S_{v'} S_{w'}^*) = \sigma_\mu(S_{v'} S_{w'}^* \alpha_H^{i\beta}(S_v S_w^*)). \tag{2.5}$$

For this, let $v = (v_0 \dots v_{m-1})$, $w = (w_0 \dots w_{n-1})$, $v' = (v'_0 \dots v'_{p-1})$ and $w' = (w'_0 \dots w'_{q-1})$, for $m, n, p, q \in \mathbb{N}$. Note that by Proposition 2.1 (2), $S_w^* S_{v'} = 0$ if and only if either $[w] \subset [v']$ or $[w] \supset [v']$. We only consider the first case, that is $p \leq n$ and $v' = (w_0 \dots w_{p-1})$, and remark that the second case follows by exactly the same means. By Proposition 2.1 (2), we then have

$$S_v S_w^* S_{v'} S_{w'}^* = S_v S_{w_p \dots w_{n-1}}^* S_{w'}^*.$$

For $S_v S_w^* S_{v'} S_{w'}^* \in \mathcal{C}(\Sigma_A)$, it immediately follows that $v = w' w_p \dots w_{n-1}$ and that $S_v S_w^* S_{v'} S_{w'}^* = \mathbf{1}_{[v]}$. Hence, $\sigma_\mu(S_v S_w^* S_{v'} S_{w'}^*) = \mu([v])$. For the right hand side of (2.5) we have

$$\begin{aligned} S_{v'} S_{w'}^* \alpha_H^{i\beta}(S_v S_w^*) &= S_{v'} S_{w'}^* e^{-\beta H \sum_{k=0}^{m-1} H \circ \theta^k} S_v S_w^* e^{\beta H \sum_{k=0}^{n-1} H \circ \theta^k} \\ &= \mathbf{1}_{[w]} e^{-\beta \sum_{k=0}^{m-1} H \circ \theta^k \circ \tau_v \circ \theta^n + \beta \sum_{k=0}^{n-1} H \circ \theta^k}. \end{aligned}$$

In particular, it hence follows $S_{v'} S_{w'}^* \alpha_H^{i\beta}(S_v S_w^*) \in \mathcal{C}(\Sigma_A)$. The latter calculation also shows $\sigma_\mu(S_v S_w^* S_{v'} S_{w'}^*) \neq 0$ if and only if $\sigma_\mu(S_{v'} S_{w'}^* \alpha_H^{i\beta}(S_v S_w^*)) \neq 0$. Since

$[v] = \tau_v \circ \theta^n[w]$ and since $d\mu \circ \theta / d\mu = e^{\beta H}$, it follows

$$\begin{aligned}
\sigma_\mu(S_v S_w^* S_{v'} S_{w'}^*) &= \int \mathbf{1}_{[v]} d\mu = \int \mathbf{1}_{[w]} \frac{d\mu \circ \tau_v \circ \theta^n}{d\mu} d\mu \\
&= \int \mathbf{1}_{[w]} \frac{d\mu \circ \tau_v}{d\mu} \circ \theta^n \cdot \frac{d\mu \circ \theta^n}{d\mu} d\mu \\
&= \int \mathbf{1}_{[w]} e^{-\beta H \sum_{k=0}^{m-1} H \circ \theta^k \circ \tau_v \circ \theta^n} \cdot e^{\beta H \sum_{k=0}^{n-1} H \circ \theta^k} d\mu \\
&= \sigma_\mu(S_{v'} S_{w'}^* \alpha_H^{i\beta}(S_v S_w^*)).
\end{aligned}$$

This gives the identity in (2.5). To finish the proof, let $H \geq 0$ and let $\sigma_1, \sigma_2 \in \mathcal{S}_0$ such that $\Theta(\sigma_1) = \Theta(\sigma_2)$. Clearly, by definition we then have that $\sigma_1|_{\mathcal{C}(\Sigma_A)} = m_{\sigma_1} = m_{\sigma_2} = \sigma_2|_{\mathcal{C}(\Sigma_A)}$. Furthermore, the gauge invariance in Fact 8 gives that $\sigma_1(X) = \sigma_2(X) = 0$, for all X in the closure of the vector space generated by $\{S_v S_w^* \mid v \in \mathcal{W}^m, w \in \mathcal{W}^n, m \neq n\}$. This shows that the restriction of Θ to $\mathcal{S}_0$ is a bijection. □

3. Radon–Nikodym representations of Cuntz–Krieger algebras

In this section we consider representations of Cuntz–Krieger algebras which are induced by Markov fibred systems. These representations are given by operators which act on some L^2 space and which are determined by the Radon–Nikodym derivative of the measure associated with the given Markov fibred system. We begin with the definition of a Markov fibred system (see [2]). Note that a Markov fibred system is often also referred to as a Markov map (see e.g. [1]).

Definition 3.1. For a Polish space Ω and a finite Borel measure m on Ω, let $\theta : \Omega \to \Omega$ be locally invertible non–singular transformation. A *Markov partition* is a countable partition α of Ω for which the following holds. $\theta : a \to \theta(a)$ is invertible, $\theta(a)$ is contained in the σ–algebra generated by α for all $a \in \alpha$, and the σ–algebra generated by $\{\alpha^n := \bigvee_{k=0}^{n-1} \theta^{-k}(\alpha) \mid n \in \mathbb{N}\}$ coincides with the Borel σ–algebra associated with Ω, up to sets of measure zero. If α is a Markov partition, then the system $(\Omega, m, \theta, \alpha)$ is called a *Markov fibred system*.

Note that the restriction of θ^n to some arbitrary $a \in \alpha^n$ is clearly also always invertible and non–singular, for each $n \in \mathbb{N}$. In analogy to the previous section, let τ_a refer to the inverse branch of θ^n restricted to $a \in \alpha^n$. Throughout, we will always assume that the Radon–Nikodym derivative $\frac{dm \circ \theta}{dm}$ has a continuous version on the support of m. Also, if $\alpha = \{a_i \mid i \in I\}$ is a finite partition then let $A = (a_{ij})_{i,j \in I}$ be the finite incidence matrix arising from θ (see e.g. [14]). To each generator S_i of the Cuntz–Krieger algebra $\mathcal{O}_A$ we associate an operator s_i on $L^2(\Omega, m)$ as follows.

Definition 3.2. Let $(\Omega, m, \theta, \alpha)$ be a Markov fibred system with associated incidence matrix $A = (a_{ij})_{i,j\in I}$. For each $i \in I$, let $s_i : L^2(\Omega, m) \to L^2(\Omega, m)$ be defined by

$$s_i : f \mapsto \mathbf{1}_{a_i} \cdot \left(\tfrac{dm\circ\theta}{dm}\right)^{\frac{1}{2}} \cdot f \circ \theta.$$

The C^*–operator algebra $\mathcal{R}_A(m)$ on the Hilbert space $L^2(\Omega, m)$ generated by the set $\{s_i \mid i \in I\}$ will be referred to as the *Radon–Nikodym representation* of $\mathcal{O}_A$ induced by $(\Omega, m, \theta, \alpha)$.

In order to see that a Radon–Nikodym representation of $\mathcal{O}_A$ induced by a Markov fibred system is in fact a representation of the Cuntz–Krieger algebra $\mathcal{O}_A$, we first make the following observations.

Lemma 3.3. *For each $i \in I$, we have that the operator s_i is well–defined, and that its adjoint operator $s_i^* : L^2(\Omega, m) \to L^2(\Omega, m)$ is given by*

$$s_i^* : f \mapsto \mathbf{1}_{\theta(a_i)} \cdot \left(\frac{dm\circ\tau_{a_i}}{dm}\right)^{\frac{1}{2}} \cdot f \circ \tau_{a_i}.$$

Furthermore, for each $i \in I$ and $f \in L^2(\Omega, m)$ we have

$$s_i s_i^*(f) = \mathbf{1}_{a_i}\, f \ \text{ and } \ s_i^* s_i(f) = \mathbf{1}_{\theta(a_i)}\, f.$$

Proof. First note that since m is non-singular, we have $(dm \circ \theta/dm)(\omega) > 0$, for m–almost every $\omega \in \Omega$. For each $f \in L^2(\Omega, m)$ and $i \in I$, we then have

$$(s_i(f), s_i(f)) = \int \overline{s_i(f)} s_i(f) dm = \int \mathbf{1}_{\theta(a_i)} |f|^2 dm < \infty.$$

This shows that s_i is well-defined, and also that $s_i^* s_i(f) = \mathbf{1}_{\theta(a_i)} \cdot f$. Next note that for $f, g \in L^2(\Omega, m)$,

$$\begin{aligned}
\int \overline{f}\; s_i(g)\; dm &= \int \overline{f(\omega)}\; \mathbf{1}_{a_i}(\omega)\; \left(\tfrac{dm\circ\theta}{dm}(\omega)\right)^{\frac{1}{2}}\; g\circ\theta(\omega)\; dm(\omega) \\
&= \int \overline{f\circ\tau_{a_i}(\omega)}\; \mathbf{1}_{a_i}\circ\tau_{a_i}(\omega)\; \left(\tfrac{dm\circ\theta}{dm}(\tau_{a_i}(\omega))\right)^{-\frac{1}{2}}\; g(\omega)\; dm(\omega) \\
&= \int \overline{\left(\mathbf{1}_{\theta(a_i)}(\omega)\; \left(\frac{d(m\circ\tau_{a_i})}{dm}(\omega)\right)^{\frac{1}{2}}\; f\circ\tau_{a_i}(\omega)\right)}\; g(\omega)\; dm(\omega).
\end{aligned}$$

This shows that $s_i^*(f) = \mathbf{1}_{\theta(a_i)} \cdot \left(\frac{dm\circ\tau_{a_i}}{dm}\right)^{\frac{1}{2}} \cdot f \circ \tau_{a_i}$. The remaining part of the lemma can now be obtained as follows.

$$\begin{aligned} (s_i s_i^*(f))(\omega) &= \left(s_i\left(\mathbf{1}_{\theta(a_i)} \cdot \left(\frac{dm\circ\tau_{a_i}}{dm}\right)^{\frac{1}{2}} \cdot f \circ \tau_{a_i}\right)\right)(\omega) \\ &= \mathbf{1}_{a_i}(\omega) \cdot \left(\frac{dm\circ\theta}{dm}(\omega)\right)^{\frac{1}{2}} \cdot \\ &\quad\ \mathbf{1}_{\theta(a_i)}(\theta(\omega)) \cdot \left(\frac{dm\circ\tau_{a_i}}{dm}(\theta(\omega))\right)^{\frac{1}{2}} \cdot f \circ \tau_{a_i}(\theta(\omega)) \\ &= \mathbf{1}_{a_i}(\omega) \cdot f(\omega). \end{aligned}$$

□

Corollary 3.1. *For the generators $\{s_i \mid i \in I\}$ of the Radon–Nikodym representation $\mathcal{R}_A(m)$ of a Cuntz–Krieger algebra $\mathcal{O}_A$ induced by a Markov fibred system, we have for each $i \in I$,*

$$\sum_{j\in I} s_j s_j^* = 1, \quad \textit{and } s_i^* s_i = \sum_{j\in I} a_{ij} s_j s_j^*.$$

Proof. The stated relations are immediate consequences of Definition 3.2 and Lemma 3.3. □

The following theorem summarises the results of this section.

Theorem 3.4. *Let $\mathcal{R}_A(m)$ be the Radon–Nikodym representation of the Cuntz–Krieger algebra $\mathcal{O}_A$ induced by a Markov fibred system $(\Omega, m, \theta, \alpha)$. If the incidence matrix A is irreducible, then $\mathcal{O}_A$ is $*$–isomorphic to $\mathcal{R}_A(m)$.*

Proof. Let ρ refer to the canonical $*$–homomorphism from $\mathcal{O}_A$ to $\mathcal{R}_A(m)$ such that $\rho(S_i) = s_i$, for all $i \in I$. By Corollary 3.1 we have that the generators $\{s_i\}$ of $\mathcal{R}_A(m)$ satisfy the same type of relations as the generators $\{S_i\}$ of $\mathcal{O}_A$ (see (2.1)). Since $*$–homomorphisms are continuous, one immediately obtains that ρ is surjective. In order to show injectivity, note that the kernel of ρ is a two-sided closed ideal in $\mathcal{O}_A$. Since $\mathcal{O}_A$ is simple, the assertion follows. □

4. Lyapunov spectra for KMS states on Cuntz–Krieger algebras associated with Kleinian groups

In this section we apply the results of the previous sections to a particular class of potential functions which were essential for the multifractal analysis of limit sets of Kleinian groups in [22]. We show that this type of multifractal analysis gives rise to interesting results concerning the existence of KMS states on Cuntz–Krieger algebras.

Recall that a Kleinian group is a discrete subgroup of the group of orientation preserving isometries of hyperbolic $(n+1)$–space $\mathbb{D}^{n+1}$ (see e.g. [5]). A non-elementary Kleinian groups G is called essentially free if G has a Poincaré polyhedron F (see [27]) with finitely many faces $\{f_1, \ldots, f_{2\mathfrak{g}}\}$ such that if two faces f_i and f_j intersect inside $\mathbb{D}^{n+1}$, then the two associated generators g_i and g_j of G commute. As a consequence of Poincaré's theorem (see [15]), we therefore have that for an essentially free Kleinian group there are no relations other than those arising from cusps of rank greater than 1. Furthermore, recall that to each essentially free Kleinian group G we can associate the following expansive coding map T, an analogue of the Bowen–Series map in higher dimensions ([9], [38]). For e_i denoting the image of the projection of f_i from some fixed chosen point in F to the boundary $\mathbb{S}^n$ of hyperbolic space, let α be the partition of $L_r(G)$ generated by $\{e_1 \cap L_r(G), \ldots, e_{2\mathfrak{g}} \cap L_r(G)\}$. To each $a = e_{i_1} \cap \cdots \cap e_{i_k} \cap L_r(G) \in \alpha$ we then associate some arbitrary fixed $j(a) \in \{i_1, \ldots, i_k\}$. With $L_r(G)$ referring to the radial limit set, that is the intersection of $L(G)$ with the complement of the set of parabolic fixed points of G, we define

$$T : L_r(G) \to L_r(G), \quad T|_a := g_{j(a)} \text{ for } a \in \alpha.$$

For further details on the construction of this map, we refer to [38]. As shown in [38], the system $(L_r(G), \nu, T, \alpha)$ is a Markov fibred system, for which T is topologically mixing, as well as conservative and ergodic with respect to the canonical T-invariant measure ν in the measure class of the Patterson measure associated with G (for the construction of the Patterson measure we refer to [30], [32], [39]). It hence follows that the incidence matrix A arising from the symbolic dynamics of T is irreducible. Also, we clearly have that there exists a canonical coding map $\pi : (\Sigma_A, \theta) \to (L(G), T)$ for which $\pi \circ \theta = T \circ \pi$, and which is one–one on $L_r(G)$. We now introduce the relevant potential functions. Namely, let J be the continuous potential function which is given by

$$J : \Sigma_A \to \mathbb{R}, w \mapsto \log |T'(\pi(w))|.$$

Moreover, we define for $s \in \mathbb{R}$,

$$J_s := sJ + P(-sJ) \text{ and } I_s := J_s + \chi - \chi \circ \theta, \tag{4.1}$$

where $P(-sJ)$ refers to the pressure function of $-sJ \in \mathcal{C}(\Sigma_A)$, and $\chi \in \mathcal{C}(\Sigma_A)$ is determined by $\mathcal{L}_{-I_s}\mathbf{1} = \mathbf{1}$. Note that if there are no parabolic elements then e^{χ} is the unique eigenfunction of $\mathcal{L}_{-J_s}$ associated with the eigenvalue 1. In case there are parabolic elements the significance of χ is slightly more involved and will be given in the proof of the following theorem. Finally, recall that the Hausdorff dimension

$\dim_H(\mu)$ of a Borel measure μ on $\mathbb{R}^n$ is given by

$$\dim_H(\mu) := \inf\{\dim_H(E) : E \text{ is a Borel set with } \mu(E) > 0\}.$$

Theorem 4.1. *Let $\mathcal{O}_A$ be the Cuntz–Krieger algebra associated with an essentially free Kleinian group G. Then there is a maximal interval $(\alpha_-, \alpha_+) \subset \mathbb{R}_+$ and a real analytic function $s : (\alpha_-, \alpha_+) \to \mathbb{R}$ such that for each $\alpha \in (\alpha_-, \alpha_+)$ there exists a unique $(I_{s(\alpha)}, 1)$–KMS state $\sigma_{s(\alpha)}$ on $\mathcal{O}_A$, for which we in particular have*

$$\sigma_{s(\alpha)}(J) = \alpha. \tag{4.2}$$

Furthermore, $\sigma_{s(\alpha)}|_{\mathcal{C}(\Sigma_A)}$ is a θ–invariant weak $(-I_{s(\alpha)})$–Gibbs measure, and for each $\alpha \in (\alpha_-, \alpha_+)$ we have

$$\dim_H\left(\sigma_{s(\alpha)}|_{\mathcal{C}(\Sigma_A)} \circ \pi^{-1}\right) = \frac{\sigma_{s(\alpha)}(J_{s(\alpha)})}{\sigma_{s(\alpha)}(J)}. \tag{4.3}$$

Additionally, if G has no parabolic elements then in the above statements the open interval (α_-, α_+) can be replaced by the closed interval $[\alpha_-, \alpha_+]$. (For a further discussion of the boundary points α_- and α_+, we refer to Remark (1) below).

Proof. First note that since in (4.2) and (4.3) only restrictions of KMS-states to $\mathcal{C}(\Sigma_A)$ are considered, it is sufficient to verify (4.2) and (4.3) for certain fixed points of the dual Perron–Frobenius operator associated with some suitable potential function. Next, recall that in [22] we studied fractal measures and Hausdorff dimensions of the α-level sets

$$M_\alpha := \left\{\xi \in L(G) \mid \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log|T'(T^k(\xi))| = \alpha\right\}, \text{ for } \alpha \in \mathbb{R}.$$

Let us first discuss the case in which G has no parabolic elements. In this case the map T is expanding and $\log|T'| > 0$, and we begin with computing the Lyapunov spectrum associated with the 'homological growth rate', that is $\alpha \mapsto \dim_H(M_\alpha)$. Note that the following differs from the approach in [22], and hence gives an alternative proof of the results of [22] for the case in which G has no parabolic elements. Also, in the following we assume that the reader is familiar with the basic results in multifractal analysis (see e.g. [13], [18], [33]).

From a purely algebraic point of view the presumably most obvious way to establish a KMS state on the noncommutative algebra $\mathcal{O}_A$ associated with G is provided by the measure of maximal entropy m arising from θ. Clearly, the topological entropy of (Σ_A, θ) is $h_{\text{top}} := \log(2\mathfrak{g} - 1)$, and hence the potential giving rise to m is the constant function $\phi :\equiv \log(2\mathfrak{g} - 1)$. From this we immediately obtain that here the relevant

gauge action is the one-parameter group of $*$–automorphisms $\left(\alpha_\phi^t\right)$, given by

$$\alpha_\phi^t(S_j) := e^{it\phi}\, S_j = (2\mathfrak{g}-1)^{it}\, S_j \text{ for all } j \in \{1,\ldots,2\mathfrak{g}\}, t \in \mathbb{R}.$$

In order to determine the associated KMS state, we compute the temperature at which the system is at equilibrium. Observe that for the topological pressure P of the system we have for $s \in \mathbb{R}$,

$$\begin{aligned} P(-s\phi) &= \lim_{n\to\infty} \frac{1}{n} \log\left(2\mathfrak{g}(2\mathfrak{g}-1)^{n-1} e^{-snh_{\mathrm{top}}}\right) \\ &= (1-s)h_{\mathrm{top}}. \end{aligned}$$

This shows that $P(-s\phi) = 0$ if and only if $s = 1$, and hence the system is at equilibrium exactly for the inverse temperature $\beta = 1$. Therefore, the KMS state canonically associated with ϕ is the $(\phi, 1)$–KMS state σ_ϕ. Note that by Fact 9 we in particular have that $\sigma_\phi|_{\mathcal{C}(\Sigma_A)} = m$. Also, one immediately verifies that m is a θ-invariant $(-\phi)$–Gibbs measure for the Hölder continuous potential ϕ. Therefore, we can apply standard multifractal analysis, which gives that for the level–sets

$$F_\beta := \left\{ w \in \Sigma_A : \lim_{r\to 0} \frac{\log m\left(\pi^{-1}(B(\pi(w), r))\right)}{\log r} = \beta \right\}$$

we have

$$\dim_H(\pi(F_{\beta(q)})) = s(q) + q\beta(q).$$

In here, the function $s : \mathbb{R} \to \mathbb{R}$ is determined by the equation $P(-s(q)J - q\phi) = 0$, and the function β is given by $\beta(q) := -s'(q)$. Also, we have that the function given by $\beta(q) \mapsto \dim_H(\pi(F_{\beta(q)}))$ is real analytic on the image of β, which is a closed interval $[\beta_-, \beta_+]$. For the θ–invariant, ergodic I_s–Gibbs measure $m_{s(q)}$ (see (4.1)) one then immediately verifies

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \phi \circ \theta^k(w)}{\sum_{k=0}^{n-1} J \circ \theta^k(w)} = \beta(q), \text{ for } m_{s(q)}\text{–almost every } w \in \Sigma_A.$$

We now make the following observation, where we have set $\alpha(q) := \frac{h_{\mathrm{top}}}{\beta(q)}$.

$$\begin{aligned} \dim_H(\pi(F_{\beta(q)})) &= \dim_H\left(\pi\left(\left\{ w \in \Sigma_A : \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \phi\circ\theta^k(w)}{\sum_{k=0}^{n-1} J\circ\theta^k(w)} = \beta(q) \right\}\right)\right) \\ &= \dim_H\left(\pi\left(\left\{ w \in \Sigma_A : \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} J\circ\theta^k(w)}{n} = \alpha(q) \right\}\right)\right). \end{aligned}$$

Note that this shows in particular that in the absence of parabolic elements the multifractal spectrum of the measure of maximal entropy coincides with the Hausdorff dimension spectrum of the homological growth rates considered in [22]. Summarising

the above in terms of α, we now have

$$\dim_H(m_{s(q)} \circ \pi^{-1}) = s(q) + q\frac{h_{\mathrm{top}}}{\alpha(q)},$$

where $s(q)$ is given by $P(-s(q)J - q\phi) = 0$, or what is equivalent $P(-s(q)J) = qh_{\mathrm{top}}$. Using $P'(-s(q)J)s'(q) = h_{\mathrm{top}}$, we can now proceed as follows. First note that, by the above,

$$\alpha(q) = \frac{h_{\mathrm{top}}}{\beta(q)} = \frac{h_{\mathrm{top}}}{-s'(q)} = -P'(-s(q)J) = \int J \, dm_{s(q)}, \tag{4.4}$$

and that the function given by $\alpha(q) \mapsto \dim_H(m_{s(q)} \circ \pi^{-1})$ is real analytic on the image of α, which is a closed interval $[\alpha_-, \alpha_+]$. Furthermore, we have

$$\begin{aligned} \dim_H(m_{s(q)} \circ \pi^{-1}) &= \frac{s(q)\alpha(q) + P(-s(q)J)}{\alpha(q)} \\ &= \frac{-s(q)P'(-s(q)J) + P(-s(q)J)}{\alpha(q)} \\ &= \frac{\int \left(s(q)J + P(-s(q)J)\right) \, dm_{s(q)}}{\int J \, dm_{s(q)}} \\ &= \frac{m_{s(q)}\left(J_{s(q)}\right)}{m_{s(q)}\left(J\right)}. \end{aligned} \tag{4.5}$$

We now use Fact 9 which gives that there exists an $\left(I_{s(\alpha)}, 1\right)$–KMS state $\sigma_{s(\alpha)}$ on $\mathcal{O}_A$ such that $\Theta(\sigma_{s(\alpha)}) = m_{s(\alpha)}$. Hence, by combining this with (4.4) and (4.5), the statements in (4.2) and (4.3) follow. The fact that $\sigma_{s(\alpha)}$ is unique can be seen as follows. Fact 9 states that Θ is injective if the underlying potential is strictly positive. Therefore, it is sufficient to show that $I_{s(\alpha)} > 0$. This follows, since $\mathcal{L}_{-I_{s(\alpha)}}\mathbf{1} = \mathbf{1}$ and hence,

$$\sum_{u \in \theta^{-1}(\{w\})} e^{-I_{s(\alpha)}(u)} = 1, \text{ for all } w \in \Sigma_A. \tag{4.6}$$

Finally, note that since $-I_{s(\alpha)}$ is Hölder continuous, the Perron-Frobenius-Ruelle Theorem (see e.g. [7]) implies that $\mathrm{Fix}\left(\mathcal{L}^*_{-I_{s(\alpha)}}\right)$ is a singleton. This finishes the proof if G has no parabolic elements.

We now consider the parabolic situation. Hence, let G be an essentially free Kleinian group with parabolic elements. It is well–known that in this case the limit set $L(G)$ can be written as the disjoint union of $L_r(G)$ and the countable set of fixed points of the parabolic transformations in G (see [6]). In contrast to the previous case, T is now expansive and the function $\log |T'|$ is equal to zero precisely on the fixed points of the parabolic generators of G. In addition to the coding by Σ_A, there is an alternative coding which is provided by the method of inducing. That is, for a

subset B of Σ_A we obtain the induced map $\widetilde{\theta} : B \to B, w \mapsto \theta^{N(w)}(w)$, where $N : \Sigma_A \to \mathbb{N} \cup \{\infty\}, w \mapsto \inf\{n \in \mathbb{N} \mid \theta^n(w) \in B\}$ (for further details we refer to the Appendix). We always assume that $\pi(B)$ is bounded away from the set of fixed points of the parabolic generators of G. This guarantees that $\widetilde{\theta}$ is an expanding Markov fibred system $(B, \widetilde{\nu}, \widetilde{\theta}, \widetilde{\alpha})$ with respect to a countable partition of B, where $\widetilde{\nu}$ refers to the pull–back under π^{-1} of the invariant version of the restriction of the Patterson measure to $\pi(B)$. For further details on the construction of this system we refer to [38]. The canonical potential function $\widetilde{J}$ for the induced system is given for $w \in B$ by

$$\widetilde{J}(w) := \sum_{k=0}^{N(w)-1} J(\theta^k(w)).$$

As shown in [22], in this parabolic situation there is a maximal interval $(\alpha_-, \alpha_+) \subset \mathbb{R}_+$ and a real analytic function $s : (\alpha_-, \alpha_+) \to \mathbb{R}$ such that for each $\alpha \in (\alpha_-, \alpha_+)$ there exists a unique $(-\widetilde{J}_{s(\alpha)})$–Gibbs measure $\widetilde{\mu}_{s(\alpha)}$, where $\widetilde{J}_{s(\alpha)} := s(\alpha)\widetilde{J}+P(-s(\alpha)J)N$. In particular, $\widetilde{\mu}_{s(\alpha)}(B \cap \pi^{-1}(M_\alpha)) = 1$ and

$$\dim_H(\widetilde{m}_{s(\alpha)} \circ \pi^{-1}) = \dim_H(M_\alpha) = \frac{\int \widetilde{J}_{s(\alpha)} d\widetilde{m}_{s(\alpha)}}{\int \widetilde{J} d\widetilde{m}_{s(\alpha)}}. \tag{4.7}$$

Moreover, with $\widetilde{h}$ referring to the eigenfunction for the eigenvalue 1 of the Perron–Frobenius operator $\widetilde{\mathcal{L}}_{-\widetilde{J}_{s(\alpha)}}$ of the induced system, let

$$\widetilde{I}_{s(\alpha)} := s(\alpha)\widetilde{J} + P(-s(\alpha)J)N + \log \widetilde{h} - \log \widetilde{h} \circ \widetilde{T}.$$

Then there exists a unique $\widetilde{\theta}$–invariant $(-\widetilde{I}_{s(\alpha)})$–Gibbs measure $\widetilde{m}_{s(\alpha)}$ in the measure class of $\widetilde{\mu}_{s(\alpha)}$. Since $\widetilde{m}_{s(\alpha)}$ is $\widetilde{\theta}$–invariant we obtain a θ–invariant measure $m_{s(\alpha)}$ by Kac's formula (see Appendix). Clearly, the measures $\widetilde{\mu}_{s(\alpha)}$, $\widetilde{m}_{s(\alpha)}$ and $m_{s(\alpha)}$ are all contained in the same measure class. By setting $H = J_{s(\alpha)}$ in Corollary 5.1, it follows that there exists a continuous function $\chi : \Sigma_A \to \mathbb{R}$ such that for the Radon–Nikodym derivative of $m_{s(\alpha)}$ we have (see Corollary 5.1 and the remark thereafter)

$$\frac{dm_{s(\alpha)} \circ \theta}{dm_{s(\alpha)}} = e^{J_{s(\alpha)}+\chi-\chi\circ\theta},$$

or equivalently $m_{s(\alpha)} \in \mathrm{Fix}\left(\mathcal{L}^*_{-I_{s(\alpha)}}\right)$, where $I_{s(\alpha)} := s(\alpha)J + P(-s(\alpha)J) + \chi - \chi \circ \theta$. Therefore, [21] implies that $m_{s(\alpha)}$ is a θ–invariant weak $(-I_{s(\alpha)})$–Gibbs measure. Next we show uniqueness of $m_{s(\alpha)}$. First note that [22, Proof of Theorem 1.2] shows that equilibrium measures for $-I_{s(\alpha)}$ are mapped to equilibrium measures for $-\widetilde{I}_{s(\alpha)}$ by inducing. If we restrict to measures having full measure on $\bigcup_{i=0}^{\infty} \theta^{-i}(B)$, then the inverse of this mapping is given by Kac's formula. Since by [29, Theorem 2.2.9] the set of equilibrium measures for $-\widetilde{I}_{s(\alpha)}$ is a singelton, it is now sufficient to

show that for every ergodic equilibrium measure ν for the potential $-I_{s(\alpha)}$ we have $\nu(\bigcup_{i=0}^{\infty}\theta^{-i}(B))=1$. Indeed, since the complement of $\bigcup_{i=0}^{\infty}\theta^{-i}B$ corresponds to the countable set of parabolic fixed points of G, we have that $\nu(\Sigma_A\setminus\bigcup_{i=0}^{\infty}\theta^{-i}B)=1$ implies $h_\nu=0=\int -s(\alpha)I\,d\nu=0$. Hence, since ν is an equilibrium state, it follows $P(-I_{s(\alpha)})=0$. This contradicts the fact that $P(-I_s)>0$ for all $s<\delta$, and therefore gives the uniqueness of $m_{s(\alpha)}$.

In order to verify the statements in (4.2) and (4.3), observe that by construction of $m_{s(\alpha)}$ we have

$$\int J dm_{s(\alpha)}=\frac{1}{\widetilde{m}_{s(\alpha)}(N)}\int \widetilde{J}d\widetilde{m}_{s(\alpha)} \tag{4.8}$$

and

$$\int J_{s(\alpha)} dm_{s(\alpha)}=\frac{1}{\widetilde{m}_{s(\alpha)}(N)}\int \widetilde{J}_{s(\alpha)}d\widetilde{m}_{s(\alpha)}. \tag{4.9}$$

Since $I_{s(\alpha)}$ is strictly positive for $\alpha\in(\alpha_-,\alpha_+)$, we can now apply Fact 9 as in the previous case. It follows that there exists a $\left(I_{s(\alpha)},1\right)$–KMS state $\sigma_{s(\alpha)}$ such that $\sigma_{s(\alpha)}|_{\mathcal{C}(\Sigma_A)}=m_{s(\alpha)}$. Hence, by combining this with (4.7), (4.8) and (4.9), the assertions in (4.2) and (4.3) follow. □

By combining Theorem 3.4 and Theorem 4.1, we immediately obtain the following result. For further details on the relations between KMS states and vector states we refer to [10].

Corollary 4.1. *Let $\mathcal{O}_A$ be the Cuntz–Krieger algebra associated with an essentially free Kleinian group G, and let $s:(\alpha_-,\alpha_+)\to\mathbb{R}$ and $m_{s(a)}$ be given by the previous theorem. Then there exists a real analytic family $\{\mathcal{R}_A(m_{s(a)})\mid a\in(\alpha_-,\alpha_+)\}$ of faithful Radon–Nikodym representations $\mathcal{R}_A(m_{s(a)})$ induced by the Markov fibred system $(L(G),m_{s(a)}\circ\pi^{-1},T,\alpha)$. In particular, for each $a\in(\alpha_-,\alpha_+)$ we have that the $(I_{s(a)},1)$–KMS state $\sigma_{s(a)}$ in the previous theorem is a vector state which is given by*

$$\sigma_{s(a)}(X)=(\mathbf{1},X(\mathbf{1}))_{s(a)}\,.$$

In here, the inner product $(\cdot,\cdot)_{s(a)}$ refers to the inner product on the Hilbert space $\left(L^2(L(G)),m_{s(a)}\circ\pi^{-1}\right)$.

Remarks.

(1) Note that α_+ is always given by $\alpha_+=\lim_{s\to-\infty}P(-sJ)/(-s)$. Similar, if G has no parabolic elements then $\alpha_-=\lim_{s\to\infty}P(-sJ)/(-s)$. Whereas, if G has parabolic elements then $\alpha_-=\lim_{s\nearrow\delta}P(-sJ)/(\delta-s)$, where $\delta=\delta(G)$ refers to the exponent of convergence of G. Here, the parabolic case has to be treated with extra care, since as we have shown in [22] in this situation a phase transition can

occur at δ (see also [38], [23], [24]). More precisely, we have the following scenario, where $k_{\max}$ refers to the maximal possible rank of the parabolic elements in G. For $\delta \leq (k_{\max}+1)/2$ we have that $\alpha_- = 0$. In this situation one immediately verifies that $\widetilde{m}_{s(0)}(N) = \infty$, and consequently Kac's formula is not applicable. However, we still obtain a θ–invariant probability measure $m_{s(0)}$ as the weak limit of a sequence $\left(m_{s(\alpha_n)}\right)$, for α_n tending to 0 from above. Note that the measure $m_{s(0)}$ has to be purely atomic. On the other hand, if $\delta > (k_{\max}+1)/2$, then $\alpha_- > 0$ and $\widetilde{m}_{s(0)}(N) < \infty$. Hence, we can argue as in the proof of Theorem 4.1 to obtain that in this case the boundary point α_- can be included in the statement of Theorem 4.1.

(2) In order to see that the assignment $q \mapsto s(q)$ gives rise to a strictly convex function, one can argue as follows. We only consider the non–parabolic case, and refer to [22] for the parabolic situation. Using the notation in the proof of Theorem 4.1, we have for the second derivative of s (see e.g. [13, p. 237]),

$$s''(q) = \frac{D_q(s'(q)J - \phi)}{\int \log|T'|\, dm_q},$$

where D_q refers to the asymptotic covariance given for a Hölder continuous function f on Σ_A by

$$D_q(f) := \sum_{k=0}^{\infty} \left(\int f \cdot f \circ \theta^k \, dm_q - \left(\int f \, dm_q \right)^2 \right).$$

Therefore, the function s is strictly convex if and only if $s(q)J + q\phi$ is not cohomologous to a constant. In order to see that the latter does in fact hold, one can argue similar as in the proofs of the 'dynamical rigidity theorems' of [8], [29] and [41]. Namely, the assumption that $s(q)J + q\phi$ is cohomologous to a constant is equivalent to the statement that there exists a constant R such that for all $n \in \mathbb{N}$ and $w \in \Sigma_A$ for which $\theta^n(w) = w$ (see e.g. [29, Theorem 2.2.7]),

$$\sum_{k=0}^{n-1} \left(s(q)J(\theta^k(w)) + q\phi(\theta^k(w)) \right) = nR.$$

One immediately verifies that the latter identity is equivalent to

$$s(q) \log |\left(T^n\right)'(\pi(w))| = n(R - q h_{\mathrm{top}}).$$

This shows that the periodic points of period n in $L(G)$ must all have equal multipliers, which is clearly absurd for the conformal system given by G. (Note that in here the constant R is in fact given by $R = P(q\phi) - P(-s(q)J)$).

(3) Finally, we remark that if G has no parabolic elements then the above analysis gives rise to the following estimate of the asymptotic growth rate of the word metric in

G for generic elements of $L(G)$. For this note that the Lyapunov spectrum of the measure of maximal entropy attains its maximum precisely at the exponent of convergence $\delta = \delta(G)$. In the notation used in the proof of Theorem 4.1, we then have $s(q) = \delta$, and hence since $P(\delta J) = 0$ and $P(-s(q)J) = qh_{\text{top}}$, it follows that $q = 0$. This implies that

$$\beta(0) = \frac{h_{\text{top}}}{\int \log |T'| \, d\nu},$$

where ν refers to the invariant version of the Patterson measure μ constructed with respect to the origin in $\mathbb{D}^{n+1}$. Now, let ξ_t refer to the unique point on the ray from the origin to $\xi \in \mathbb{S}^n$ at hyperbolic distance t to the origin. Also, let $[\xi_t]$ denote the word length of g, for $g \in G$ determined by $\xi_t \in g(F)$. Then the arguments in the proof of Theorem 4.1 immediately imply that for μ–almost every $\xi \in L(G)$ we have

$$\lim_{t \to \infty} \frac{t}{[\xi_t]} = \frac{h_{\text{top}}}{\beta(0)} = \int \log |T'| \, d\nu.$$

5. APPENDIX

In this appendix we give a refinement of a formula of Kac in the context of Markov fibred systems. We obtain explicit formulae which allow to compute the Radon–Nikodym derivative of a θ–invariant measure on the whole system (Ω, θ) in terms of the Radon–Nikodym derivative of a $\widetilde{\theta}$–invariant measure on an induced system $(B, \widetilde{\theta})$.

More precisely, let $(\Omega, m, \theta, \alpha)$ be a conservative and ergodic Markov fibred system with respect to the finite partition α of Ω. Furthermore, let $B \subset \Omega$ be measurable with respect to the σ–algebra generated by $\alpha^n := \bigvee_{k=0}^{n-1} \theta^{-k}(\alpha)$, for some $n \in \mathbb{N}$. Also, let $\widetilde{\theta}$ refer to the *induced transformation* given by

$$\widetilde{\theta} : B \to B, \omega \mapsto \theta^{N(\omega)}(\omega),$$

where

$$N : \Omega \to \mathbb{N} \cup \{\infty\}, \omega \mapsto \inf\{n \in \mathbb{N} \mid \theta^n(\omega) \in B\}.$$

It is well–known that the induced system $(B, \widetilde{m}, \widetilde{\theta}, \widetilde{\alpha})$ is again a conservative and ergodic Markov fibred system, where $\widetilde{\alpha}$ denotes the associated countable partition which can be finite or infinite, and $\widetilde{m} := m|_B$ (see e.g. [1]). The inverse branches of $\widetilde{\theta}$ will be denoted by $\widetilde{\tau}_a$, for $a \in \widetilde{\alpha}$.

Recall that θ–invariant measures and $\widetilde{\theta}$–invariant measures are related as follows. If ν is a given θ–invariant measure then we obtain a $\widetilde{\theta}$–invariant measure by restricting ν to B. Conversely, if $\widetilde{\nu}$ is a given $\widetilde{\theta}$–invariant measure such that $\widetilde{\nu}(N) < \infty$ one obtains a θ–invariant probability measure ν by the following formula of Kac (see [19]). Namely,

for $\sum_{k=0}^{N-1} f \circ \theta^k \in L^1(\widetilde{\nu})$ we have

$$\int f d\nu = \frac{1}{\widetilde{\nu}(N)} \int_B \sum_{k=0}^{N(\omega)-1} f \circ \theta^k(\omega)\, d\widetilde{\nu}(\omega).$$

We now investigate for this situation in which way the two associated Radon–Nikodym derivatives $d\nu \circ \theta / d\nu$ and $d\widetilde{\nu} \circ \widetilde{\theta} / d\widetilde{\nu}$ are related. One direction is immediately given by the chain rule. Namely, for a given θ–invariant measure ν we have

$$\log \frac{d\widetilde{\nu} \circ \widetilde{\theta}}{d\widetilde{\nu}}(\omega) = \sum_{k=0}^{N(\omega)-1} \log \left(\frac{d\nu \circ \theta}{d\nu}(\theta^k(\omega)) \right).$$

The converse direction is slightly more delicate and will be subject of the following proposition. We remark that it might be that this statement is known to experts in this area, however we were unable to find it in the literature and hence decided to include the proof. We require the following notation. Let $D_n := \{\omega \in \Omega \mid N(\omega) = n\}$, and put $N(A) := n$ if $A \subset D_n$ for some $n \in \mathbb{N}$. Also, for $A \subset D_n$ such that for some $b \in \widetilde{\alpha}$ we have that either $A \subset B \cap b$ or $A \subset \Omega \setminus B$ and $\theta^n(A) \subset b$, we define

$$\mathcal{Z}(A) := \begin{cases} \{a \in \widetilde{\alpha} \mid A \subset \theta^{N(a)}(a)\} & : \quad A \subset B \\ \{a \in \widetilde{\alpha} \mid N(a) > N(A), A \subset \theta^{N(a)-N(A)}(a)\} & : \quad A \subset \Omega \setminus B. \end{cases}$$

Furthermore, we put $\mathcal{Z}(\omega) := \mathcal{Z}(b)$ if either $\omega \in b \in \widetilde{\alpha}$ or $\omega \in \Omega \setminus B$ such that $\omega \in b \in \alpha^n$, where $\theta^n(b) \in \widetilde{\alpha}$ for some $n \in \mathbb{N}$. Note that in the first case the set $\{\widetilde{\tau}_a \mid a \in \mathcal{Z}(\omega)\}$ represents the set of inverse branches of $\widetilde{\theta}$ at ω, whereas in the second case the set $\{\widetilde{\tau}_a \mid a \in \mathcal{Z}(\omega)\}$ refers to the set of inverse branches of $\widetilde{\theta}$ at $\theta^{N(\omega)}(\omega)$ with the additional property that $\omega \in \{\theta^k(\widetilde{\tau}_a(\omega)) \mid 1 \leq k < N(\widetilde{\tau}_a(\omega))\}$, for each $a \in \mathcal{Z}(\omega)$. Hence, for $\omega \notin B$ we in particular have

$$\{\widetilde{\tau}_a(\theta^{N(\omega)}(\omega)) \mid a \in \mathcal{Z}(\omega)\} = \bigcup_{l=N(\omega)+1}^{\infty} \theta^{-(l-N(\omega))}(\omega) \cap D_l \cap B.$$

Proposition 5.1. *Let $(\Omega, m, \theta, \alpha)$ be a conservative and ergodic Markov fibred system, and let $(B, \widetilde{m}, \widetilde{\theta}, \widetilde{\alpha})$ be the induced system as introduced above. If $\widetilde{\nu}$ is a given $\widetilde{\theta}$–invariant measure which is absolutely continuous with respect to $\widetilde{m}$, then the following holds for the θ–invariant measure ν obtained through Kac's formula.*

(1) *For ν–almost all $\omega \in B$, we have*

$$\frac{d\nu \circ \theta}{d\nu}(\omega) = \left(\sum_{a \in \mathcal{Z}(\theta\omega)} \frac{d\widetilde{\nu} \circ \widetilde{\tau}_a}{d\widetilde{\nu}}(\theta^{N(\omega)}(\omega)) \right) \cdot \frac{d\widetilde{\nu} \circ \widetilde{\theta}}{d\widetilde{\nu}}(\omega).$$

(2) *For ν–almost all $\omega \in \Omega \setminus B$, we have*

$$\frac{d\nu\circ\theta}{d\nu}(\omega)=\left(\sum_{a\in\mathcal{Z}(\theta\omega)}\frac{d\widetilde{\nu}\circ\widetilde{\tau}_a}{d\widetilde{\nu}}(\theta^{N(\omega)}(\omega))\right)\Bigg/\left(\sum_{a\in\mathcal{Z}(\omega)}\frac{d\widetilde{\nu}\circ\widetilde{\tau}_a}{d\widetilde{\nu}}(\theta^{N(\omega)}(\omega))\right).$$

Proof. First note that the infinite sums in (1) and (2) do converge. This follows since the $\widetilde{\theta}$–invariance of $\widetilde{\nu}$ implies, for ν–almost all $\omega \in B$,

$$\sum_{a\in\mathcal{Z}(\omega)}\frac{d\widetilde{\nu}\circ\widetilde{\tau}_a}{d\widetilde{\nu}}(\omega)=\sum_{a\in\widetilde{\alpha}}\frac{d\widetilde{\nu}\circ\widetilde{\tau}_a}{d\widetilde{\nu}}(\omega)=1. \tag{5.1}$$

Let $A \subset D_n \setminus B$ such that $\theta^n(A) \subset a$, for some $n \in \mathbb{N}$ and $a \in \widetilde{\alpha}$. We then have

$$\begin{aligned}
\nu(A) &= \frac{1}{\widetilde{\nu}(N)}\int_B \sum_{k=0}^{N(\omega)-1}\mathbf{1}_A\circ\theta^k(\omega)\,d\widetilde{\nu}(\omega)\\
&= \frac{1}{\widetilde{\nu}(N)}\sum_{l=1}^{\infty}\int_{B\cap D_l}\sum_{k=0}^{l-1}\mathbf{1}_A\circ\theta^k(\omega)\,d\widetilde{\nu}(\omega)\\
&= \frac{1}{\widetilde{\nu}(N)}\sum_{l=1}^{\infty}\sum_{k=0}^{l-1}\widetilde{\nu}(B\cap D_l\cap\theta^{-k}(A))\\
&= \frac{1}{\widetilde{\nu}(N)}\sum_{l=n+1}^{\infty}\widetilde{\nu}(B\cap D_l\cap\theta^{-(l-n)}(A))\\
&= \frac{1}{\widetilde{\nu}(N)}\sum_{a\in\mathcal{Z}(A)}\widetilde{\nu}(\widetilde{\tau}_a(\theta^n A)).
\end{aligned}$$

Hence, we have for $\omega \in A$,

$$\frac{d\nu}{d\nu\circ\theta^n}(\omega)=\sum_{a\in\mathcal{Z}(\omega)}\frac{d\widetilde{\nu}\circ\widetilde{\tau}_a}{d\widetilde{\nu}}(\theta^n(\omega)). \tag{5.2}$$

Therefore, $\nu(\theta A) = \left(\sum_{a\in\mathcal{Z}(\theta A)}\widetilde{\nu}(\widetilde{\tau}_a(\theta^n A))\right)/\widetilde{\nu}(N)$ and

$$\frac{d\nu\circ\theta}{d\nu\circ\theta^n}(\omega)=\sum_{a\in\mathcal{Z}(\theta\omega)}\frac{d\widetilde{\nu}\circ\widetilde{\tau}_a}{d\widetilde{\nu}}(\theta^n(\omega)). \tag{5.3}$$

Combining (5.2) and (5.3), the assertion follows in the case in which $\omega \in D_n \setminus B$ for some $n > 1$. The case $\omega \in D_1 \setminus B$ is an immediate consequence of (5.1) and (5.2). This proves the assertion in (2). The proof of (1) is now an immediate consequence of

(5.3). Namely, for each $\omega \in D_n \cap B$ with $n > 1$,

$$\begin{aligned}\frac{d\nu\circ\theta}{d\nu}(\omega) &= \frac{d\nu\circ\theta}{d\nu\circ\theta^n}(\omega)\Big/\frac{d\nu}{d\nu\circ\theta^n}(\omega) = \frac{d\nu\circ\theta}{d\nu\circ\theta^n}(\omega)\cdot\frac{d\widetilde{\nu}\circ\widetilde{\theta}}{d\widetilde{\nu}}(\omega)\\ &= \left(\sum_{a\in\mathcal{Z}(\theta\omega)}\frac{d\widetilde{\nu}\circ\widetilde{\tau}_a}{d\widetilde{\nu}}(\theta^n(\omega))\right)\cdot\frac{d\widetilde{\nu}\circ\widetilde{\theta}}{d\widetilde{\nu}}(\omega).\end{aligned}$$

If $\omega \in D_1 \cap B$, then we have similar to the previous case that the statement is an immediate consequence of (5.1). □

For the following we define, for $\omega \in \Omega$,

$$Z(\omega) := \{\eta \in B \mid \theta^k(\eta) = \omega \text{ for some } 1 \le k \le N(\eta).\}$$

Corollary 5.1. *In the situation of the previous proposition assume that there exist measurable functions $H : \Omega \to \mathbb{R}$ and $\widetilde{\chi} : B \to \mathbb{R}$ such that for almost all $\omega \in B$,*

$$\frac{d\widetilde{\nu}\circ\widetilde{\theta}}{d\widetilde{\nu}}(\omega) = e^{\left(\sum_{k=0}^{N(\omega)-1} H\circ\theta^k(\omega)\right)+\log\widetilde{\chi}(\omega)-\log\widetilde{\chi}(\widetilde{\theta}(\omega))}.$$

Then there exists a function $\chi : \Omega \to \mathbb{R}$ such that for almost all $\omega \in \Omega$,

$$\frac{d\nu\circ\theta}{d\nu}(\omega) = e^{H(\omega)+\log\chi(\omega)-\log\chi(\theta(\omega))}.$$

In here, the function χ is given by

$$\chi(\omega) := \begin{cases} \widetilde{\chi}(\omega) & : \omega \in B\\ \left(\sum_{\eta\in Z(\omega)} e^{-\sum_{k=0}^{N(\eta)-N(\omega)-1} H\circ\theta^k(\eta)-\log\widetilde{\chi}(\eta)}\right)^{-1} & : \omega \notin B.\end{cases}$$

Proof. Note that $\theta^{N(\eta)-N(\omega)}(\eta) = \omega$, for each $\omega \in \Omega, \eta \in Z(\omega)$. We hence have for $\omega \notin B$,

$$\begin{aligned}\sum_{a\in\mathcal{Z}(\omega)}\frac{d\widetilde{\nu}\circ\widetilde{\tau}_a}{d\widetilde{\nu}}(\theta^{N(\omega)}(\omega)) &= \sum_{\eta\in Z(\omega)} e^{-\sum_{k=0}^{N(\eta)-1} H\circ\theta^k(\eta)-\log\widetilde{\chi}(\eta)+\log\widetilde{\chi}(\widetilde{\theta}(\eta))}\\ = \left(\sum_{\eta\in Z(\omega)} e^{-\sum_{k=0}^{N(\eta)-N(\omega)-1} H\circ\theta^k(\eta)-\log\widetilde{\chi}(\eta)}\right) & e^{-\sum_{k=0}^{N(\omega)-1} H\circ\theta^k(\omega)+\log\widetilde{\chi}(\theta^{N(\omega)}(\omega))}\\ = e^{-\log\chi(\omega)}\, e^{-\sum_{k=0}^{N(\omega)-1} H\circ\theta^k(\omega)+\log\widetilde{\chi}(\theta^{N(\omega)}(\omega))}.\end{aligned}$$

Combining this with (5.1) and Proposition 5.1, the assertion follows. □

Remark. Note that one immediately verifies, using (5.1), that the definition of χ in Corollary 5.1 can be rewritten so that only finite sums are involved. Therefore, it follows that χ is continuous whenever both H and $\widetilde{\chi}$ are continuous.

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